Sydney Technical High School

Student Name: _____

Teacher's Name: _____

Year 12
Mathematics Advanced

Trial Examination

August 2024

General Instructions

- Working time –3 hours + 10 minutes reading time
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided with this paper
- In questions **11-31**, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for careless setting out or illegible writing

Total marks: 90

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks

- Attempt Questions 11-31
- Allow about 2 hours and 45 minutes for this section

For marker's use only:

MC	11-18	19 – 24	25 – 31
/10	/30	/28	/27

Total	/95	%

Section I – Multiple Choice

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Each multiple choice question is worth 1 mark.
Clearly colour in each bubble of the correct answer on your multiple choice sheet in your answer booklet.

Question One

What is the gradient of the line with equation $2x - 3y + 1 = 0$

A) $-\frac{2}{3}$

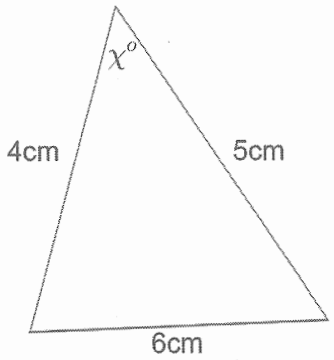
B) $-\frac{3}{2}$

C) $\frac{3}{2}$

D) $\frac{2}{3}$

Question Two

What is the area of the following triangle:



A) $Area = \frac{1}{2} \times 5 \times 4 \times \cos x$

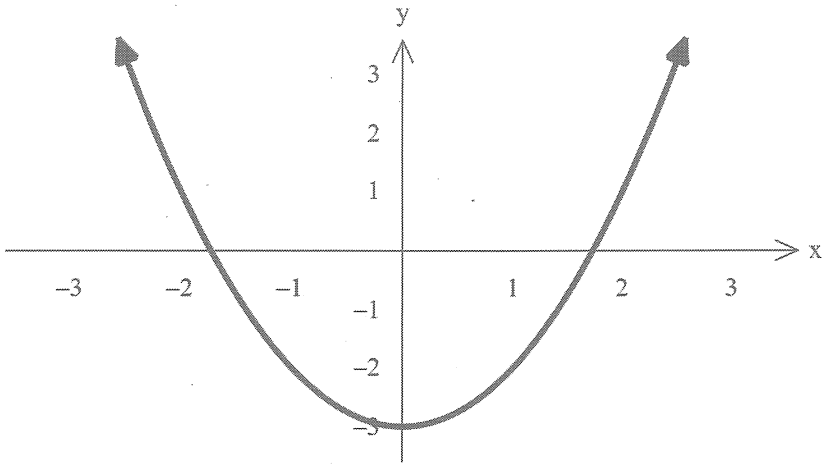
B) $Area = \frac{1}{2} \times 4 \times 6 \times \cos x$

C) $Area = \frac{1}{2} \times 5 \times 6 \times \sin x$

D) $Area = \frac{1}{2} \times 4 \times 5 \times \sin x$

Question Three

Which type of relationship best describes the curve:



A) One-to-one

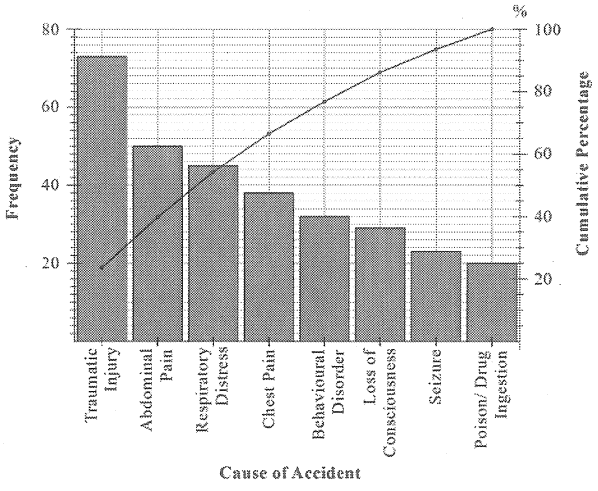
B) One-to-many

C) Many-to-one

D) Many-to-many

Question Four

The Pareto Chart below shows the reasons for an ambulance call out for a region. What percentage of patients required an ambulance call out for loss of consciousness?



A) 9%

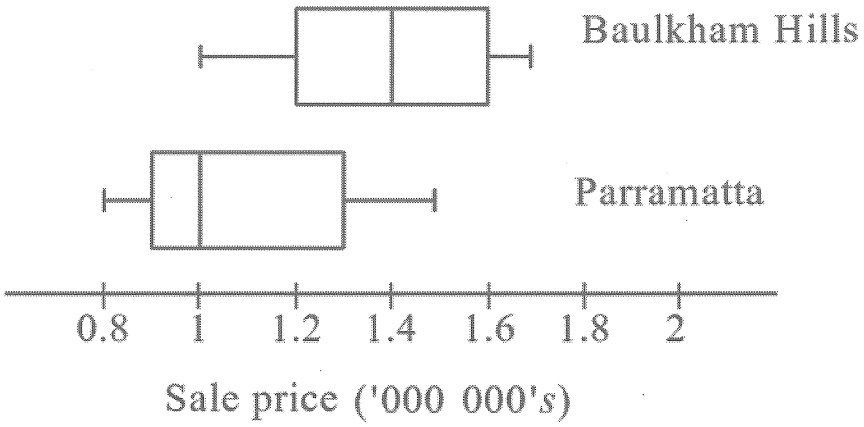
B) 24%

C) 40%

D) 86%

Question Five

The parallel box plots below compare sale prices for houses sold in Parramatta and Baulkham Hills in millions of dollars. 120 houses were sold in Baulkham Hills and 24 houses were sold in Parramatta.



Which is a true statement about the two sets of data?

A) They have the same lower quartile.

B) They have the same median.

C) They have the same range.

D) They have the same upper quartile.

Question Six

The graph of a function $y = \frac{4}{x+1}$ is translated 3 units left, and dilated vertically by a factor of $\frac{1}{2}$.

Which of the following gives the equation of the new function?

A) $\frac{y}{2} = \frac{4}{x-2}$

B) $2y = \frac{4}{x-2}$

C) $2y = \frac{4}{x+4}$

D) $\frac{y}{2} = \frac{4}{x+4}$

Question Seven

Three chess players are chosen from 3 males and 7 females.
What is the probability that they are all the same gender?

A) $\frac{1}{5}$

B) $\frac{3}{10}$

C) $\frac{2}{5}$

D) $\frac{1}{2}$

Question Eight

What is the domain of the function:

$$y = -\sqrt{1 + 3x}$$

A) $\left(-\infty, -\frac{1}{3}\right)$

B) $\left[-\frac{1}{3}, \infty\right)$

C) $\left(-\infty, -\frac{1}{3}\right]$

D) $\left[-\infty, -\frac{1}{3}\right]$

Question Nine

What is the solution of $3^x = 2$

A) $x = \frac{\log_e 2}{3}$

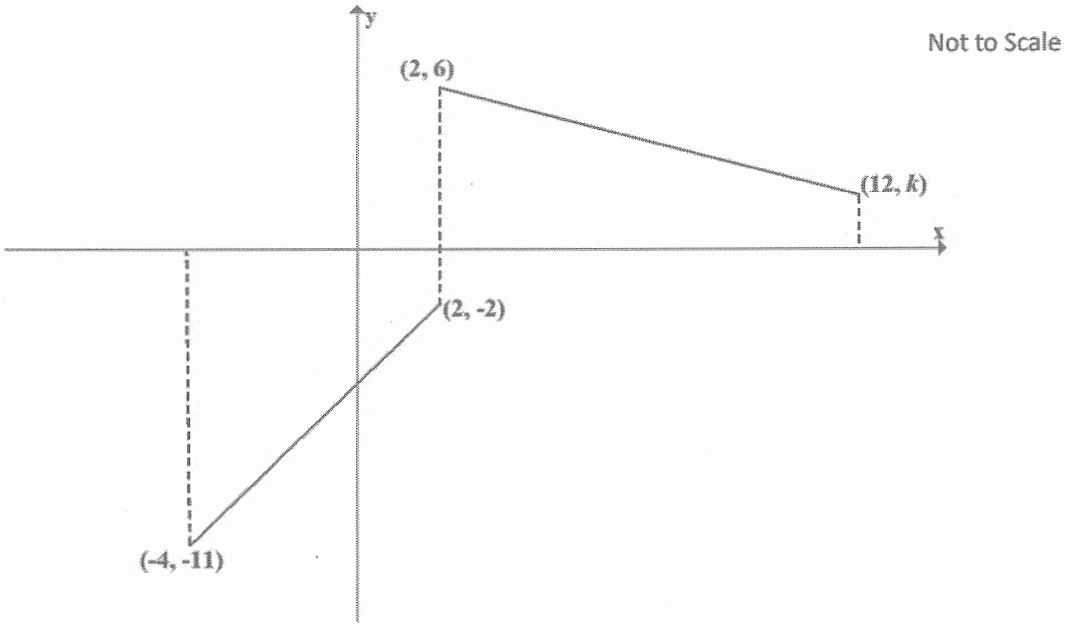
B) $x = \frac{2}{\log_e 3}$

C) $x = \log_e\left(\frac{2}{3}\right)$

D) $x = \frac{\log_e 2}{\log_e 3}$

Question Ten

The graph below shows a piecewise function $f(x)$.



Find the value of k, shown on the diagram, which satisfies $\int_{-4}^{12} f(x)dx = 0$

A) 1.8

B) 2

C) 1.55

D) 1

Section II – Short Answer

90 marks

Attempt Question 11-31

Allow about 2 hours and 45 minutes for this section

In Section II, your responses should include all relevant mathematical reasoning and/or calculations.

Question Eleven

Find values for $\tan\theta$ if $\sin\theta = \frac{2}{\sqrt{19}}$ and θ is obtuse. 2

Question Twelve

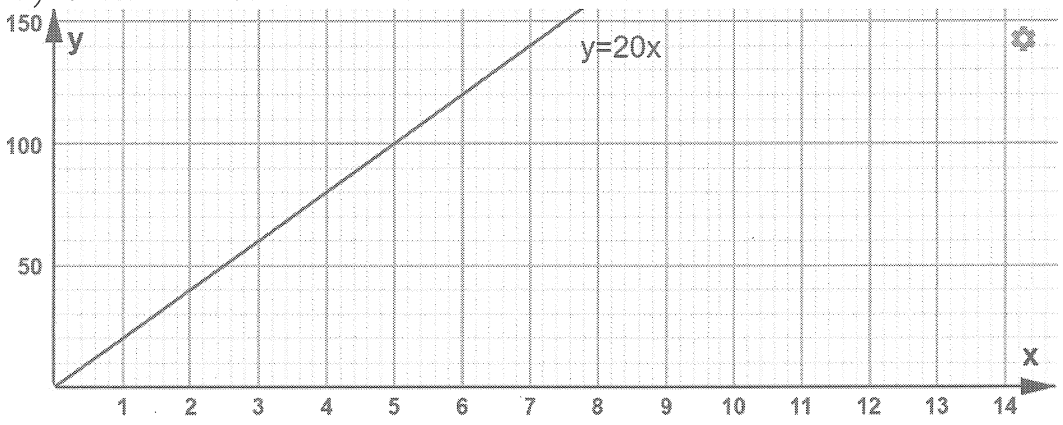
Find $\int 2 + \sin 3x \ dx$ 2

Question Thirteen

Jamie owns a company producing and selling pizzas. The income function of $y = 20x$ is shown on the graph below, where x is the number of pizzas sold. The cost of producing these pizzas includes a set up cost of \$50 and additional costs of \$10 per pizza.

a) Write the cost function in the form $y = mx + c$ 1

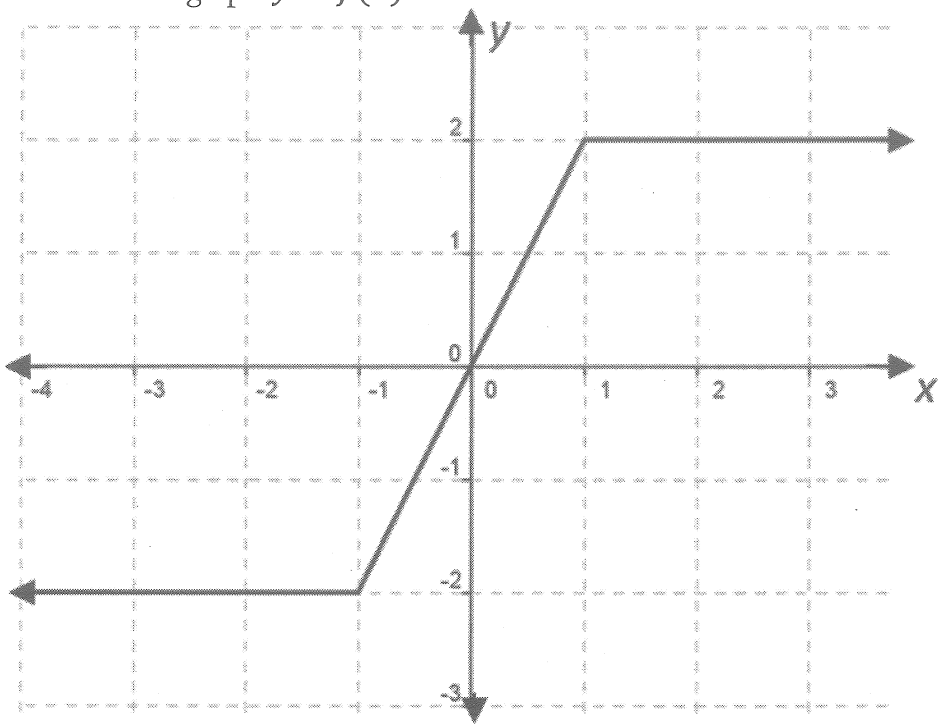
b) Sketch the cost function on the axis below: 2



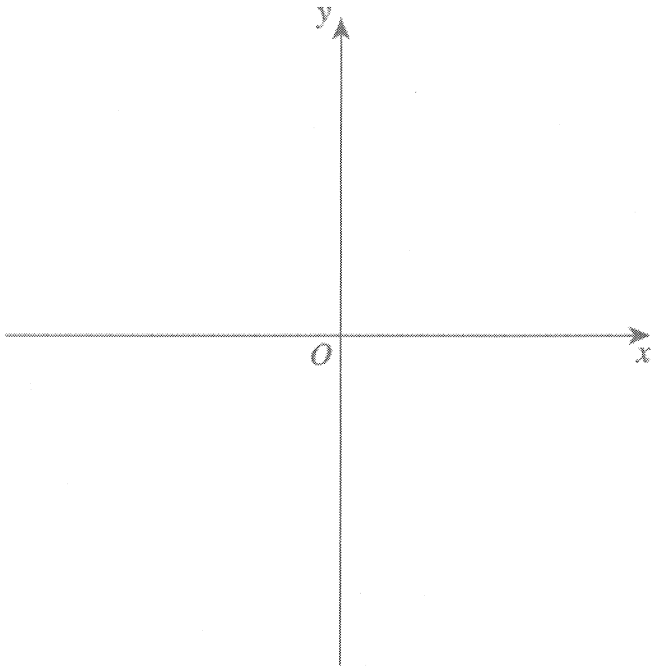
c) Hence, determine Jamie's break even point. 1

Question Fourteen

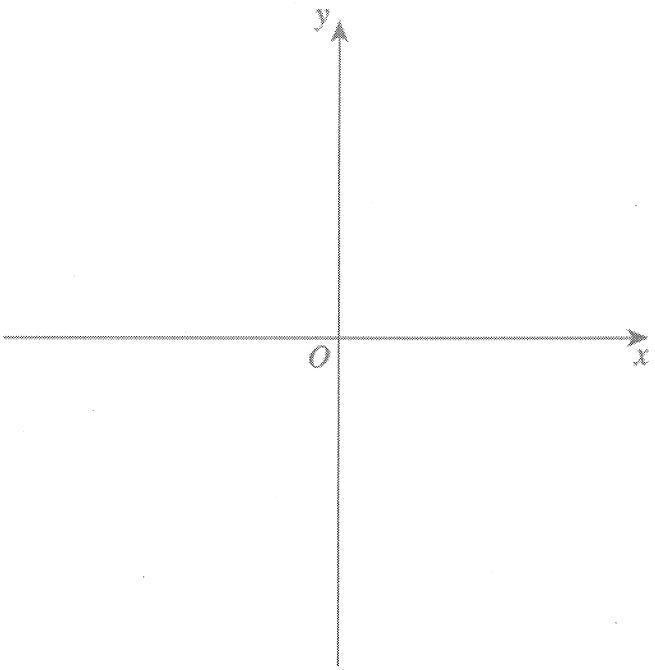
Consider the graph $y = f(x)$ shown below.



a) Sketch the graph of $y = -f(x)$ 1



b) Sketch the graph of $y = f(-x)$ 1



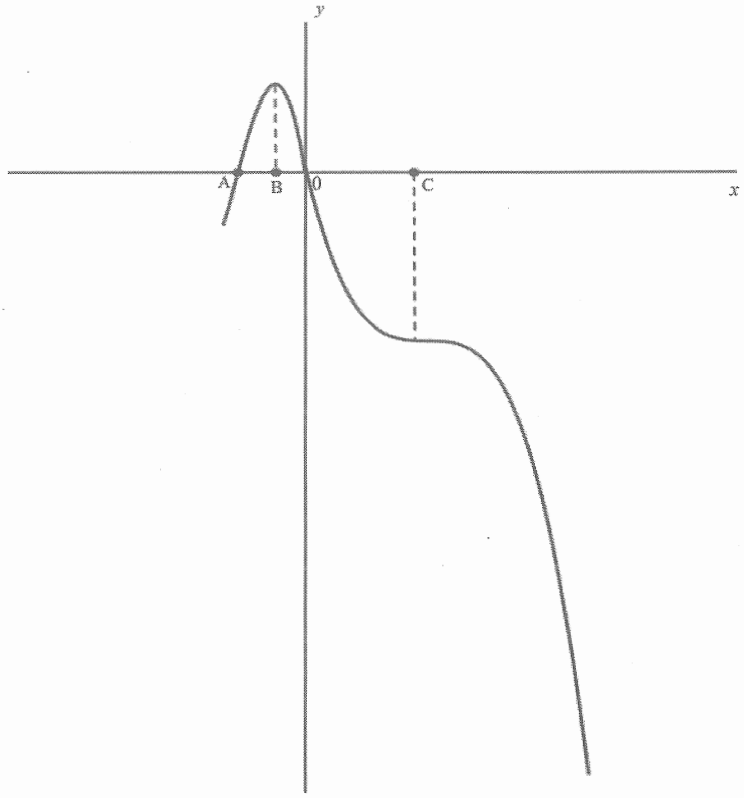
Question Fifteen

Prove that $(tanx + cotx - 1)(sinx + cosx) = \frac{secx}{cosec^2 x} + \frac{cosecx}{\sec^2 x}$ 3

Question Sixteen

The graph of $y = f(x)$ is shown below. There is a maximum turning point at $x = B$, a point of inflexion at $x = 0$, and a horizontal point of inflexion at $x = C$. 2

Draw the graph of $y = f'(x)$ on the same set of axes below.



Question Seventeen

Solve $\ln x - \frac{3}{lnx} = 2$ giving your answers in exact form. 2

Question Eighteen

Consider the quadratic function:

$$y = x^2 - x + 3$$

a) How many real roots does the function have? 2

b) What are the coordinates of the vertex? 2

c) Sketch the graph of the parabola. 2

Question Nineteen

Differentiate:

a) $y = \frac{2e}{3x^3}$ 2

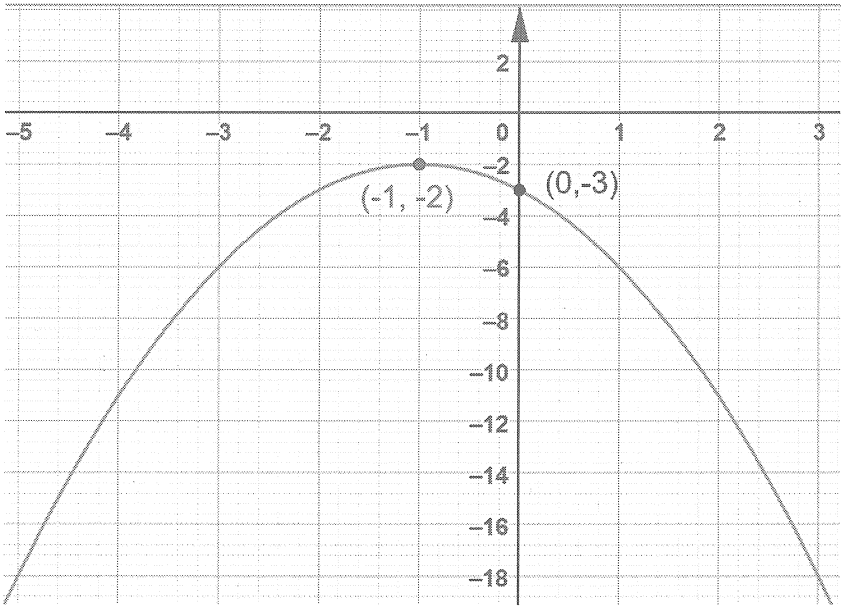
b) $f(x) = x^3 \log_e 4x$ 2

c) $y = \frac{-\ln x}{x}$ 2

d) $y = sin^3(2x + 1)$ 3

Question Twenty

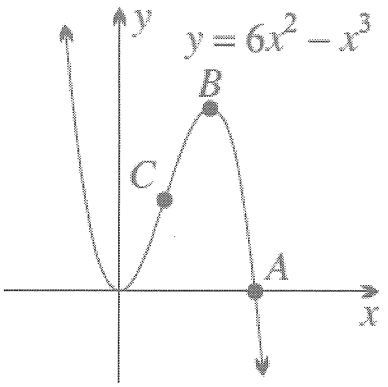
The function $f(x) = x^2$ is transformed into a new function whose graph is shown below: 2



Find the equation of the new function in the form $g(x) = k(x + b)^2 + c$ for some constants k, b, and c.

Question Twenty-One

The diagram shows a sketch of the curve $y = 6x^2 - x^3$. The curve cuts the x axis at A, has a local maximum at B, and a point of inflexion at C.



a) Find the coordinates of A. 2

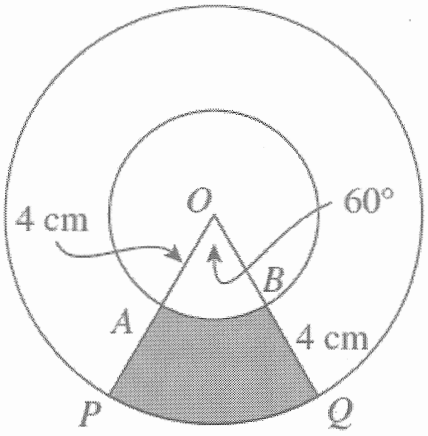
b) Find the coordinates of B. 3

c) Find the coordinates of C. 2

Question Twenty-Two

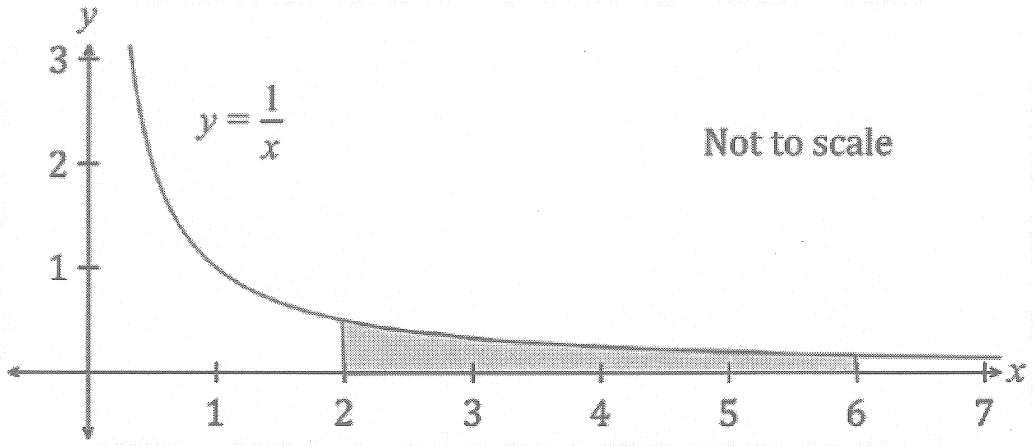
The diagram below shows two concentric circles with common centre O. 3

Find the area of the region $APQB$, leaving your answer in exact form.



Question Twenty-Three

Consider the curve $y = \frac{1}{x}$ sketched below.



a) Find the area bounded by the curve, the x-axis, and the lines $x = 2$ and $x = 6$ using the Trapezoidal Rule with five function values. Give your answer correct to three decimal places. 2

b) Calculate the same area by evaluating $\int_2^6 \frac{1}{x}\, dx$. 2
Give your answer correct to three decimal places.

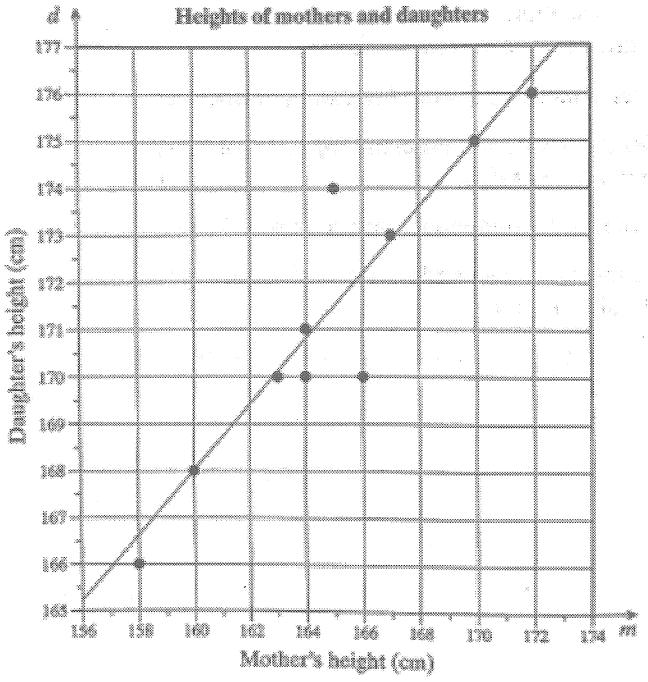
c) Explain why there is a slight difference between your answers in part (a) and part (b). 1

Question Twenty-Four

The heights (cm) of 10 mothers (m) and their daughters (d) were recorded in the table below.

Mother's height in cm (m)	170	163	160	172	164	158	164	166	167	165
Daughter's height in cm (d)	175	170	168	176	170	166	171	170	173	174

The data was used to create a scatter plot and Amelia constructed a regression line by eye, as shown below:



a) Determine the equation of Amelia's regression line. 1

b) Find the value of r, the Pearson's correlation coefficient (correct to 2 decimal places) and hence describe the relationship. 2

Question Twenty-Five

A particle with velocity, $v\ ms^{-1}$ moves along a straight line from a fixed point O

$$v = -4cost$$

where the time t is measured in seconds from $t = 0$.
It is initially 1 metre to the right of the origin.

a) When, and where, does the particle first come to rest? 2

b) Show that the total distance travelled by the particle between $0 \leq t \leq \frac{2\pi}{3}$ is $(8 - 2\sqrt{3})$ metres. 3

Question Twenty-Six

Sketch the graph of $y = 3x^4 - 4x^3 + 2$ in the space provided showing all: 5

- turning points
- points of inflexion
- y-intercept

There is no need to find the x-intercept(s).

Question Twenty-Seven

The queueing time, X minutes, of a teacher waiting on the phone with Department of Education has a probability density function

$$f(x) = \begin{cases} \frac{3}{32}x(k-x), & 0 \leq x \leq k \\ 0, & \text{otherwise} \end{cases}$$

where k is a constant.

a) Show that $k = 4$. 2

b) Find the cumulative distribution function $F(x)$. 2

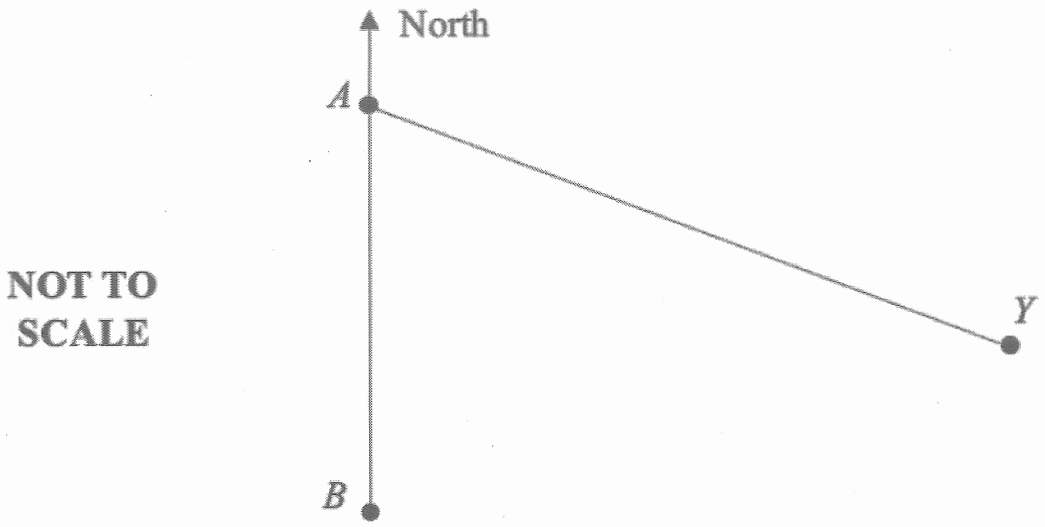
c) Find the mode of the probability density function. 1

d) Find the probability that the phone will be answered within the first minute. 1

Question Twenty-Eight

A yacht leaves Port A on a bearing of $120°$ and sails for three hours at an average speed of $15km/h$ to its destination Y where it stops.

At the same time, a speed boat also leaves from Port A and travels due south to Island B that is $30km$ from the port.



a) Calculate the distance of the yacht from Island B to the nearest kilometre. 2

b) Find the bearing of Island B from the yacht to the nearest degree. 2

c) After spending several hours at the island, the speed boat travels due north back to Port A. 2

How far south of the port will the speed boat be when it is directly west of the yacht?

Question Twenty-Nine

A small group of students were surveyed to determine whether they were part of the local baseball or tennis team. 3

- 16 people played tennis only.
- 20 people played basketball only.
- 6 people played neither.
- k people played both.

Let B be the event that a person plays baseball and let T be the event that a person plays tennis.
Determine the value of k such that the events B and T are independent.

Question Thirty

Packets of coffee beans are labelled with a net weight of 300g. It is found that the weight of a packet can be modelled by a normal distribution with mean 306g and standard deviation 3g.

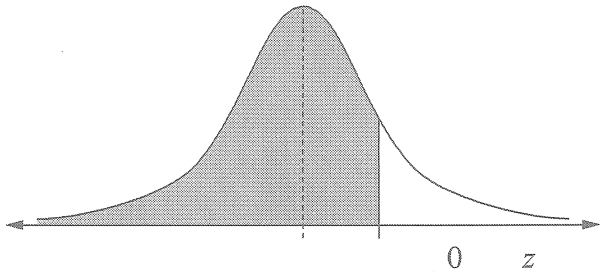
a) Use the Empirical rule to determine the probability that the weight of one packet of coffee beans is less than the advertised weight of 300g. 1

b) In a shipment of 40 boxes, each with 100 packets of coffee beans, how many packets would be expected to be underweight? 1

c) Manufacturers aim to ensure that the expected number of underweight packets in the shipment will be less than 20. The machine is adjusted to give a mean weight of 308g, with the standard deviation of 3g remaining the same. Will they meet their target? Justify your answer. 2

Use the information below to answer Question 30.

Table of values $P(Z \leq z)$ for the normal distribution $N(0, 1)$



Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995

Question Thirty-One

A prototype rocket which is initially at rest, takes off from a launchpad on the ground. It has a time of flight of T seconds, and t is the time in seconds, where $0 \leq t \leq T$.

The velocity of the rocket, $v\ ms^{-1}$, is given by:

$$v(t) = 0.5e^t \sin\left(\frac{\pi t}{10}\right)$$

a) Shortly after the rocket takes off, the engine stops and it begins to descend towards the ground.
Find the time at which the rocket begins to descend. 2

Sydney Technical High School

Student Name: Solutions

Teacher's Name: _____

Year 12
Mathematics Advanced

Trial Examination

August 2024

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MC	11-19	20 – 26	27 – 31
/10	/32	/34	/24

Total	/100	%

Section I – Multiple Choice

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Each multiple choice question is worth 1 mark.
Clearly colour in each bubble of the correct answer on your multiple choice sheet in your answer booklet.

Question One

What is the gradient of the line with equation $2x - 3y + 1 = 0$

A) $-\frac{2}{3}$

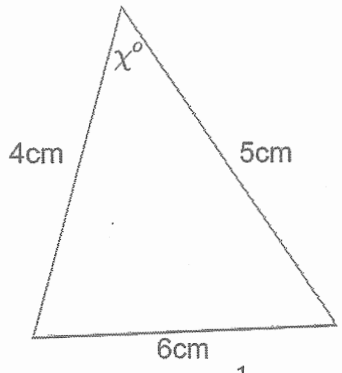
B) $-\frac{3}{2}$

C) $\frac{3}{2}$

D) $\frac{2}{3}$

Question Two

What is the area of the following triangle:



A) $Area = \frac{1}{2} \times 5 \times 4 \times \cos x$

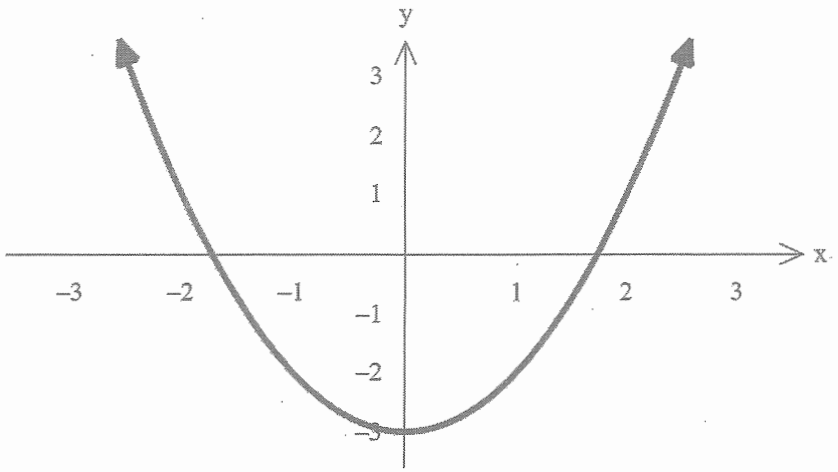
B) $Area = \frac{1}{2} \times 4 \times 6 \times \cos x$

C) $Area = \frac{1}{2} \times 5 \times 6 \times \sin x$

D) $Area = \frac{1}{2} \times 4 \times 5 \times \sin x$

Question Three

Which type of relationship best describes the curve:



A) One-to-one

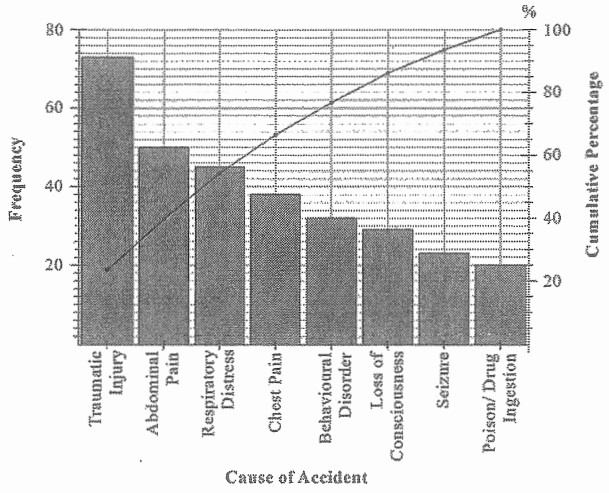
B) One-to-many

C) Many-to-one

D) Many-to-many

Question Four

The Pareto Chart below shows the reasons for an ambulance call out for a region. What percentage of patients required an ambulance call out for loss of consciousness?



A) 9%

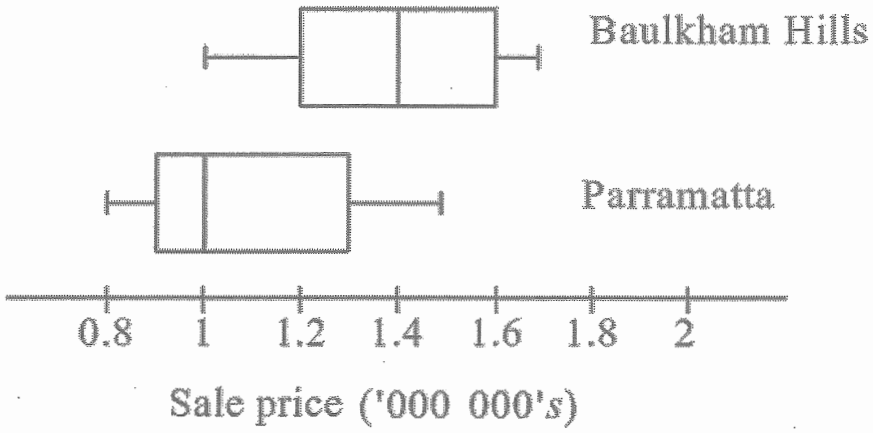
B) 24%

C) 40%

D) 86%

Question Five

The parallel box plots below compare sale prices for houses sold in Parramatta and Baulkham Hills in millions of dollars. 120 houses were sold in Baulkham Hills and 24 houses were sold in Parramatta.



Which is a true statement about the two sets of data?

A) They have the same lower quartile.

B) They have the same median.

C) They have the same range.

D) They have the same upper quartile.

Question Six

The graph of a function $y = \frac{4}{x+1}$ is translated 3 units left, and dilated vertically by a factor of $\frac{1}{2}$.

Which of the following gives the equation of the new function?

A) $\frac{y}{2} = \frac{4}{x-2}$

B) $2y = \frac{4}{x-2}$

C) $2y = \frac{4}{x+4}$

D) $\frac{y}{2} = \frac{4}{x+4}$

Question Seven

Three chess players are chosen from 3 males and 7 females.
What is the probability that they are all the same gender?

A) $\frac{1}{5}$

B) $\frac{3}{10}$

C) $\frac{2}{5}$

D) $\frac{1}{2}$

Question Eight

What is the domain of the function:

$$y = -\sqrt{1 + 3x}$$

A) $\left(-\infty, -\frac{1}{3}\right)$

B) $\left[-\frac{1}{3}, \infty\right)$

C) $\left(-\infty, -\frac{1}{3}\right]$

D) $\left[-\infty, -\frac{1}{3}\right]$

Question Nine

What is the solution of $3^x = 2$

A) $x = \frac{\log_e 2}{3}$

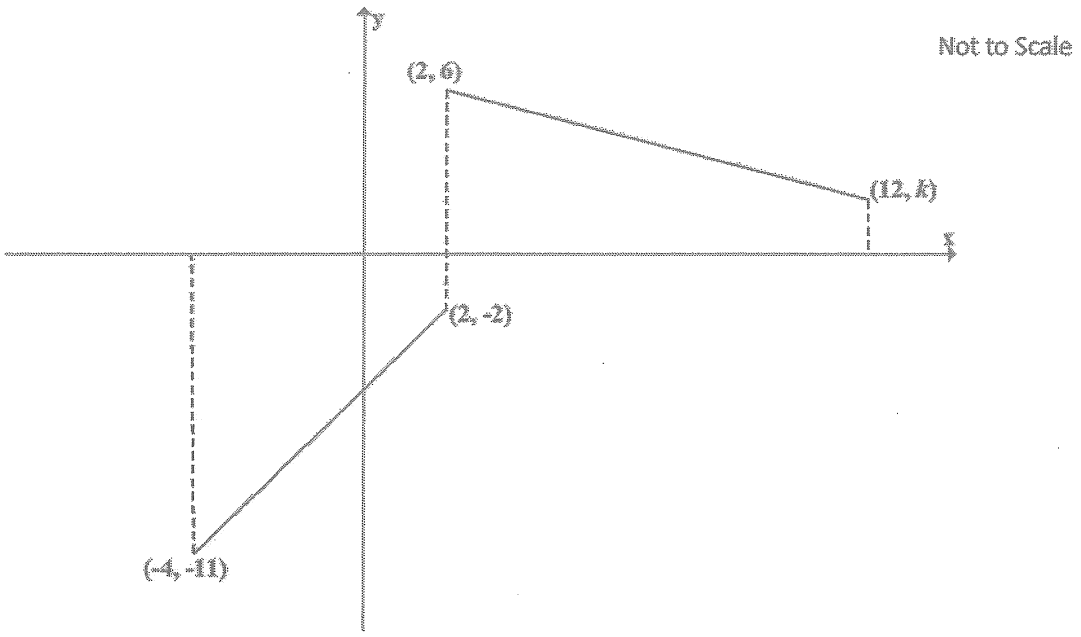
B) $x = \frac{2}{\log_e 3}$

C) $x = \log_e\left(\frac{2}{3}\right)$

D) $x = \frac{\log_e 2}{\log_e 3}$

Question Ten

The graph below shows a piecewise function $f(x)$.



Find the value of k, shown on the diagram, which satisfies $\int_{-4}^{12} f(x)dx = 0$

A) 1.8

B) 2

C) 1.55

D) 1

Section II – Short Answer

90 marks

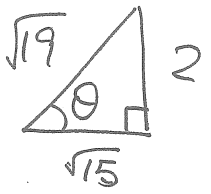
Attempt Question 11-31

Allow about 2 hours and 45 minutes for this section

In Section II, your responses should include all relevant mathematical reasoning and/or calculations.

Question Eleven

Find values for $\tan\theta$ if $\sin\theta = \frac{2}{\sqrt{19}}$ and θ is obtuse. 2



$\tan\theta = -\frac{2}{\sqrt{15}}$ OR $-\frac{2\sqrt{15}}{15}$

Question Twelve

Find $\int 2 + \sin 3x \; dx$ 2

$= 2x + \left(\frac{1}{3}\right)(-\cos 3x) + C$

$= 2x - \frac{1}{3}\cos 3x + C$

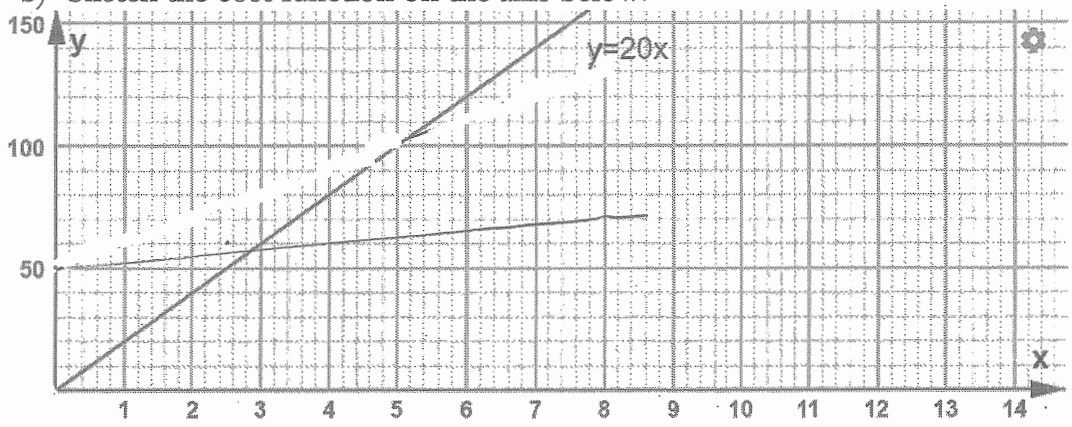
Question Thirteen

Jamie owns a company producing and selling pizzas. The income function of $y = 20x$ is shown on the graph below, where x is the number of pizzas sold. The cost of producing these pizzas includes a set up cost of \$50 and additional costs of \$2 per pizza.

a) Write the cost function in the form $y = mx + c$ 1

$y = 2x + 50$

b) Sketch the cost function on the axis below: 2



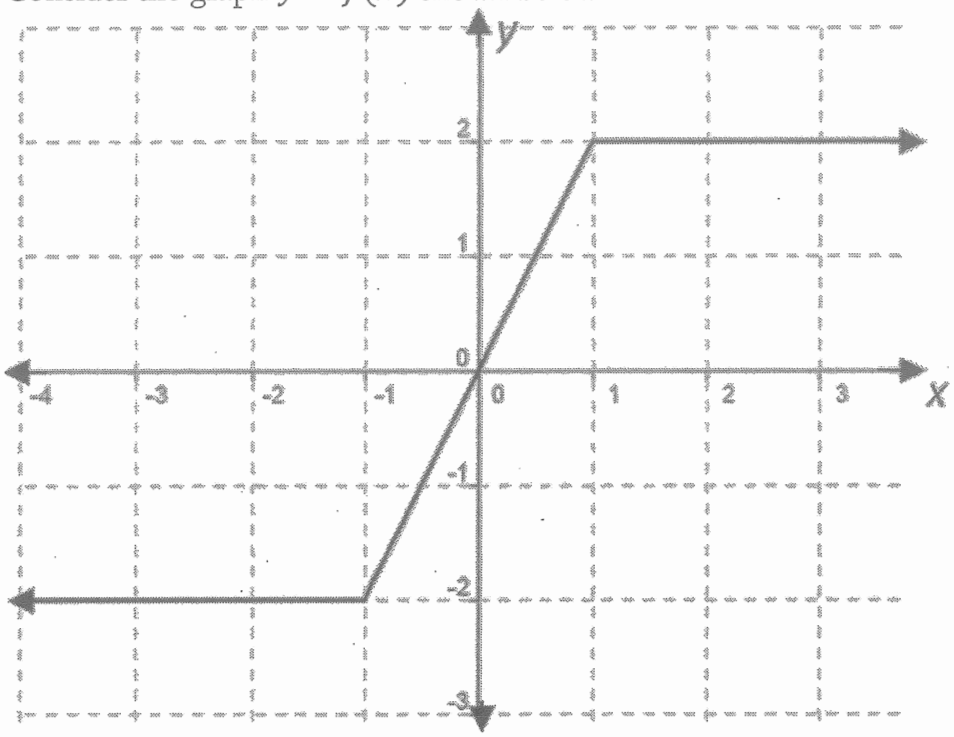
c) Hence, determine Jamie's break even point. 1

2.83 pizzas

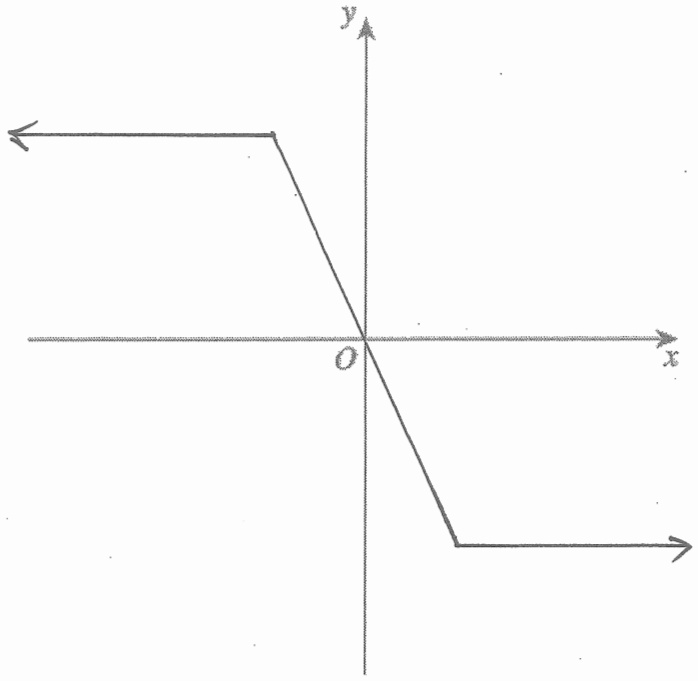
OR 3 pizzas

Question Fourteen

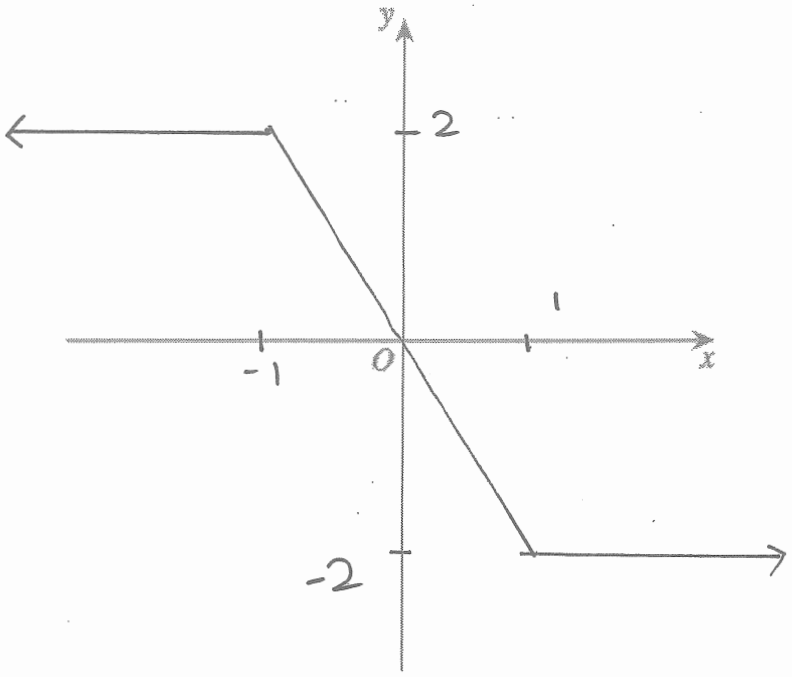
Consider the graph $y = f(x)$ shown below.



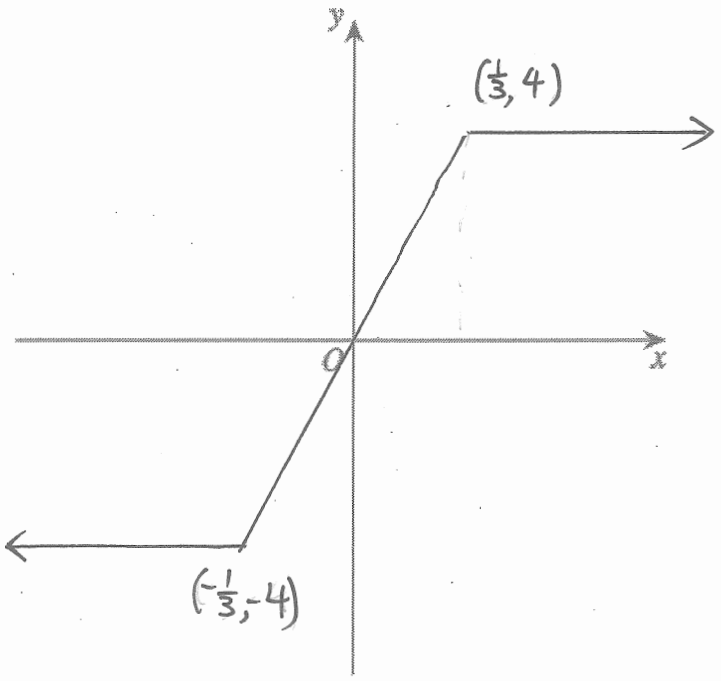
a) Sketch the graph of $y = -f(x)$ 1



b) Sketch the graph of $y = f(-x)$ 1



c) Sketch the graph of $y = 2f(3x)$ 1



Question Fifteen

Prove that $(tanx + cotx - 1)(sinx + cosx) = \frac{secx}{cosec^2x} + \frac{cosecx}{sec^2 x}$ 3

$$LHS = \frac{\sin^2 x}{\cos x} + \sin x + \frac{\sin x}{\tan x} + \frac{\cos x}{\tan x} - \sin x - \cos x$$

$$= \frac{\sin^2 x}{\cos x} + \sin x + \cos x + \frac{\cos^2 x}{\sin x} - \sin x - \cos x$$

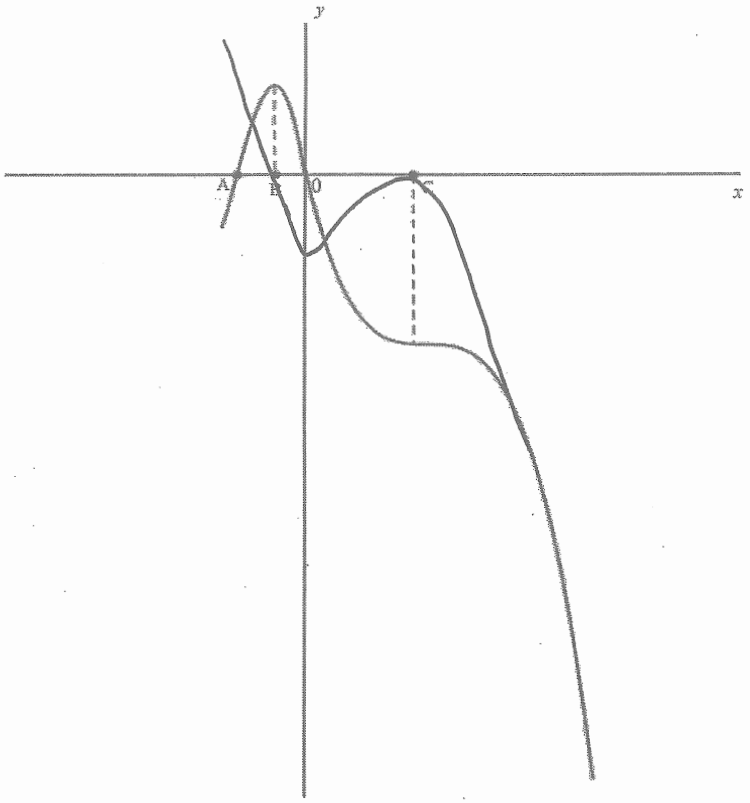
$$= \frac{\sec x}{\operatorname{cosec}^2 x} + \frac{\operatorname{cosec} x}{\sec^2 x}$$

$$= RHS$$

Question Sixteen

The graph of $y = f(x)$ is shown below. There is a maximum turning point at $x = B$, a point of inflexion at $x = 0$, and a horizontal point of inflexion at $x = C$. 2

Draw the graph of $y = f'(x)$ on the same set of axes below.



Question Seventeen

Solve $\ln x - \frac{3}{lnx} = 2$ giving your answers in exact form. 3

$(\ln x)^2 - 3 = 2\ln x \quad , \quad x > 0$

$(\ln x)^2 - 2\ln x - 3 = 0$

$(\ln x - 3)(\ln x + 1) = 0$

$\ln x = 3 \quad \text{OR} \quad \ln x = -1$

$x = e^3 \quad \text{OR} \quad x = e^{-1}$

Question Eighteen

Consider the quadratic function:

$$y = x^2 - x + 3$$

a) How many real roots does the function have? 2

$\Delta = (-1)^2 - 4(1)(3)$

$= -11 < 0$

So No real root

b) What are the coordinates of the vertex? 2

$-\frac{b}{2a} = \frac{1}{2}$

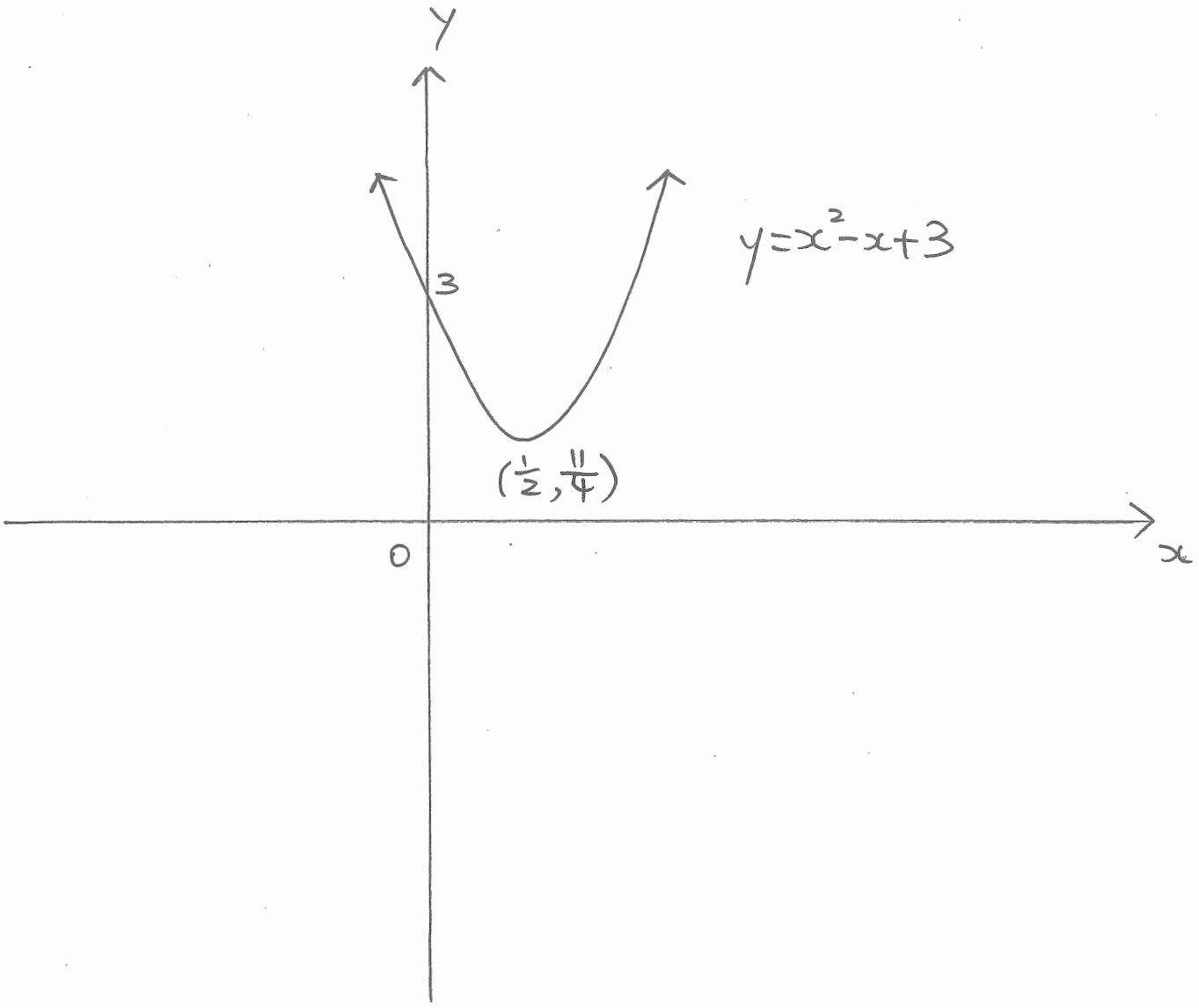
Put $x = \frac{1}{2}$ into $y = x^2 - x + 3$

$= (\frac{1}{2})^2 - \frac{1}{2} + 3$

$= \frac{11}{4}$

So the vertex is $(\frac{1}{2}, \frac{11}{4})$

c) Sketch the graph of the parabola. 2



Question Nineteen

Differentiate:

a) $y = \frac{2e}{3x^3}$ 1

$y = \frac{2e}{3} x^{-3}$

$\frac{dy}{dx} = -2ex^{-4}$ OR $\frac{-2e}{x^4}$

b) $f(x) = x^3 \log_e 4x$ 2

$f'(x) = 3x^2 \ln 4x + x^3 \left(\frac{4}{4x}\right)$

$= 3x^2 \ln 4x + x^2$

c) $y = \frac{-\ln x}{x}$ 2

$\frac{dy}{dx} = \frac{x\left(-\frac{1}{x}\right) - (-\ln x)}{x^2}$

$= \frac{\ln x - 1}{x^2}$

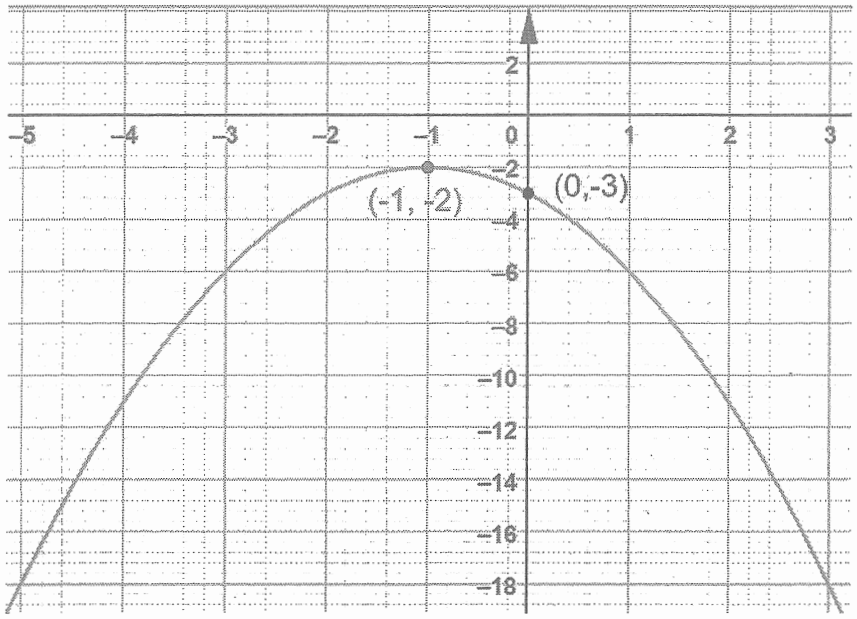
d) $y = sin^3(2x + 1)$ 2

$\frac{dy}{dx} = \left(3\sin^2(2x+1)\right)\left(2\cos(2x+1)\right)$

$= 6\sin^2(2x+1)\cos(2x+1)$

Question Twenty

The function $f(x) = x^2$ is transformed into a new function whose graph is shown below: 3



Find the equation of the new function in the form $g(x) = k(x + b)^2 + c$ for some constants k, b, and c.

$g(x) = k(x+1)^2 - 2$

$g(0) = -3$

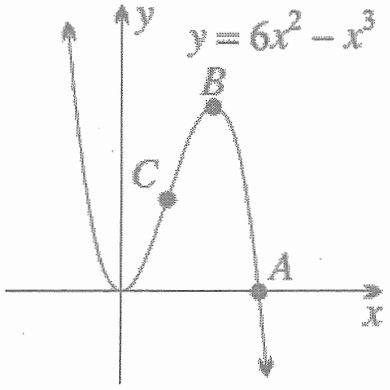
$k(1)^2 - 2 = -3$

$k = -1$

So $g(x) = -(x+1)^2 - 2$

Question Twenty-One

The diagram shows a sketch of the curve $y = 6x^2 - x^3$. The curve cuts the x axis at A, has a local maximum at B, and a point of inflexion at C.



a) Find the coordinates of A. 2

Put $y=0$, $0 = 6x^2 - x^3$

$0 = x^2(6-x)$

A is $(6,0)$

b) Find the coordinates of B. 3

B is a stationary point. $\frac{dy}{dx} = 12x - 3x^2$

At a stationary point, $\frac{dy}{dx} = 0$

So $12x - 3x^2 = 0$

$3x(4-x) = 0$

At B, $x=4$, $y=32$

So B is $(4, 32)$

c) Find the coordinates of C. 2

C is a point of inflexion

$\frac{d^2y}{dx^2} = 12 - 6x$

At a point of inflexion, $\frac{d^2y}{dx^2} = 0$

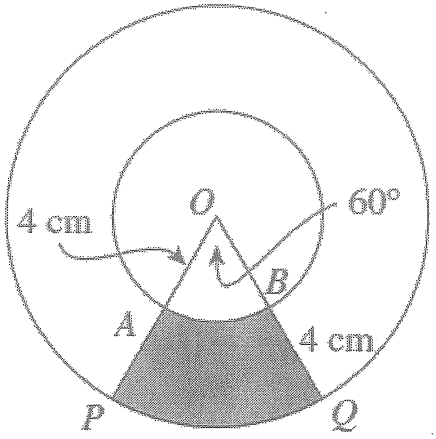
$x = 2$

So C is $(2, 16)$

Question Twenty-Two

The diagram below shows two concentric circles with common centre O. 3

Find the area of the region $APQB$, leaving your answer in exact form.



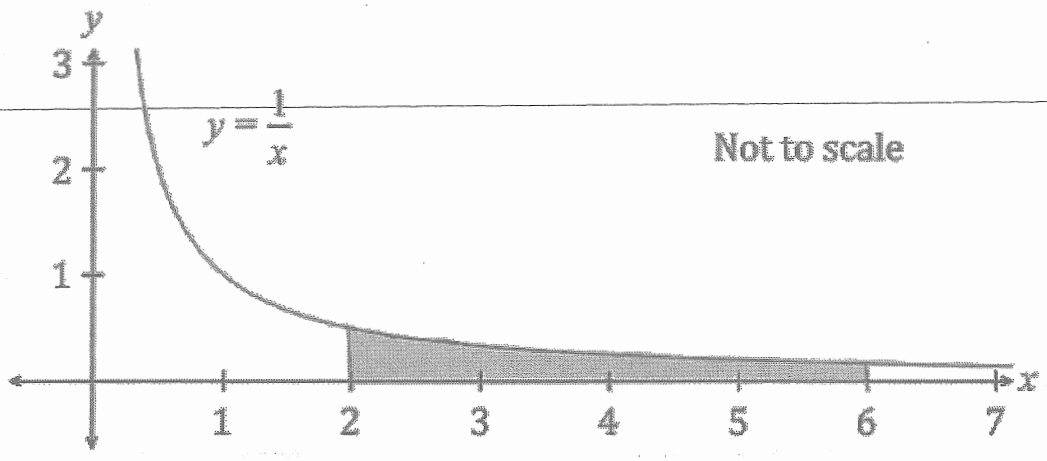
Area of the minor sector $OAB = \pi(4^2) \times \frac{60}{360} = \frac{8}{3}\pi \text{ cm}^2$

Area of the minor sector $OPQ = \pi(8^2) \times \frac{60}{360} = \frac{32}{3}\pi \text{ cm}^2$

So Area of $APQB = 8\pi \text{ cm}^2$

Question Twenty-Three

Consider the curve $y = \frac{1}{x}$ sketched below.



a) Find the area bounded by the curve, the x-axis, and the lines $x = 2$ and $x = 6$ using the Trapezoidal Rule with five function values. Give your answer correct to three decimal places. 2

Area $\doteq \frac{6-2}{2\times4}\{f(2) + f(6) + 2[f(3)+f(4)+f(5)]\}$

$\doteq \frac{4}{8}[\frac{1}{2} + \frac{1}{6} + 2(\frac{1}{3} + \frac{1}{4} + \frac{1}{5})]$

$\doteq 1.117$ unit^2

b) Calculate the same area by evaluating $\int_2^6 \frac{1}{x}\,dx$. 2
Give your answer correct to three decimal places.

$\int_2^6 \frac{1}{x}\,dx = [\ln x]_2^6$

$\doteq 1.099$ unit^2

c) Explain why the Trapezoidal Rule overestimates the area. 1

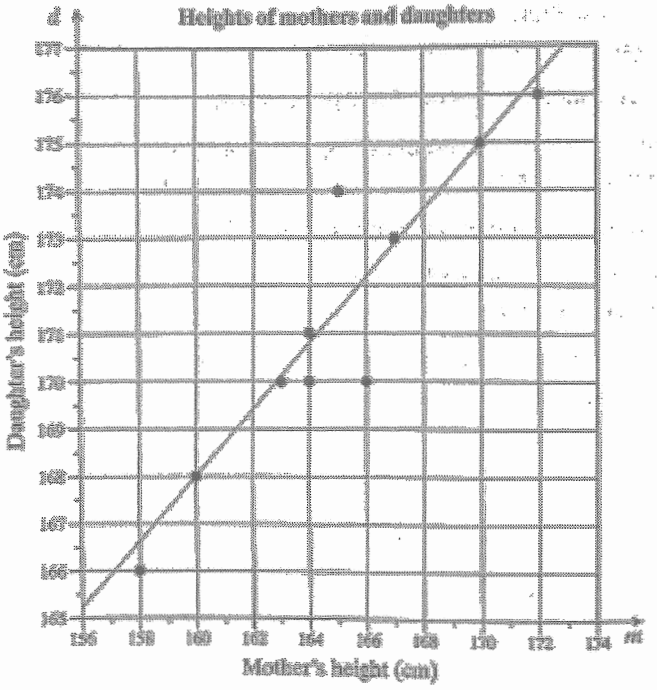
The curve is concave up

Question Twenty-Four

The heights (cm) of 10 mothers (m) and their daughters (d) were recorded in the table below.

Mother's height in cm (m)	170	163	160	172	164	158	164	166	167	165
Daughter's height in cm (d)	175	170	168	176	170	166	171	170	173	174

The data was used to create a scatter plot and Amelia constructed a regression line by eye, as shown below:



a) Determine the equation of Amelia's regression line. 1

$y = \frac{7}{10}x + 56$

b) Find the value of r, the Pearson's correlation coefficient (correct to 2 decimal places) and hence describe the relationship. 2

$r \doteq 0.93$

Very strong positive correlation

Question Twenty-Five

A particle with velocity, $v\ ms^{-1}$ moves along a straight line from a fixed point O

$$v = -4cost$$

where the time t is measured in seconds from $t = 0$.
It is initially 1 metre to the right of the origin.

a) When, and where, does the particle first come to rest? 3

$-4\cos t = 0$

$\cos t = 0$

$t = \frac{\pi}{2}s$

$x = \int -4\cos t\, dt$

$x = -4\sin t + C$

$1 = 0 + C$

$x = -4\sin t + 1$

At $t = \frac{\pi}{2}$, $x = -3$ where is 3 metres to the left of the origin.

b) Show that the total distance travelled by the particle between $0 \leq t \leq \frac{2\pi}{3}$ is $(8 - 2\sqrt{3})$ metres. 3

$$\int_0^{\frac{\pi}{2}} -4\cos t \, dt$$

$$= [-4\sin t]_0^{\frac{\pi}{2}}$$

$$= -4$$

So distance travelled between $0 \leq t \leq \frac{\pi}{2}$ is 4 metres.

$$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} -4\cos t \, dt$$

$$= [-4\sin t]_{\frac{\pi}{2}}^{\frac{2\pi}{3}}$$

$$= -4\left(\frac{\sqrt{3}}{2} - 1\right)$$

$$= 4 - 2\sqrt{3}$$

So the total distance travelled between $0 \leq t \leq \frac{2\pi}{3}$ is $(8 - 2\sqrt{3})$ metres.

Question Twenty-Six

Sketch the graph of $y = 3x^4 - 4x^3 + 2$ in the space provided showing all:

- turning points
- points of inflexion
- y-intercept

4

There is no need to find the x-intercept(s).

$\frac{dy}{dx} = 12x^3 - 12x^2$ $\quad \frac{d^2y}{dx^2} = 36x^2 - 24x$

At turning points, $\frac{dy}{dx} = 0$

$12x^3 - 12x^2 = 0$

$12x^2(x-1) = 0$

$x = 0$ or $x = 1$

At $x = 0$, $\frac{d^2y}{dx^2} = 0$

At $x = 1$, $\frac{d^2y}{dx^2} = 12 > 0$ so $(1, 1)$ is a minimum point.

At points of inflexion, $\frac{d^2y}{dx^2} = 0$

$36x^2 - 24x = 0$

$12x(3x-2) = 0$

$x = 0$ or $x = \frac{2}{3}$

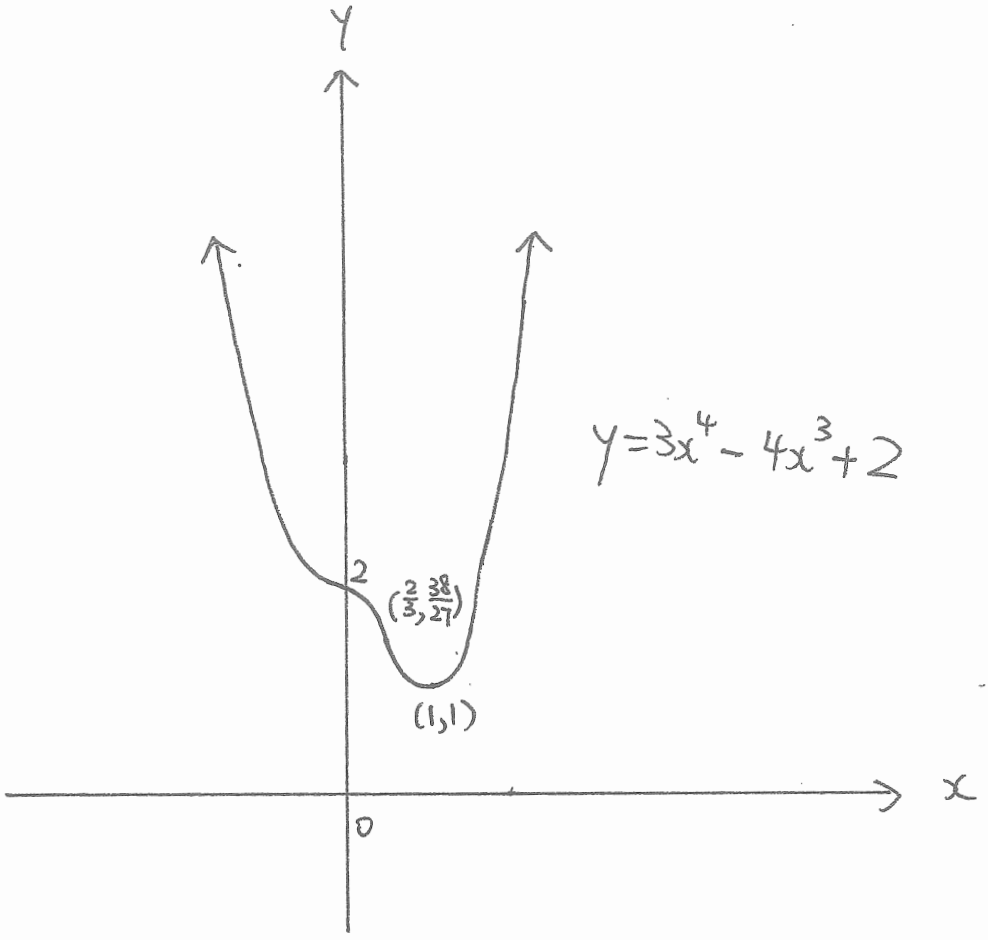
Let's test the concavity

x	-1	0	$\frac{1}{2}$	$\frac{2}{3}$	1
$\frac{d^2y}{dx^2}$	60	0	-3	0	12
Concavity	$\cup$		$\cap$		$\cup$

So $(0, 2)$ is a horizontal point of inflexion

$(\frac{2}{3}, \frac{38}{27})$ is a point of inflexion.

The y-intercept is $(0, 2)$


y
x
0
2
$(\frac{2}{3}, \frac{38}{27})$
(1,1)
$y=3x^4-4x^3+2$

Question Twenty-Seven

The queueing time, X minutes, of a teacher waiting on the phone with Department of Education has a probability density function

$$f(x) = \begin{cases} \frac{3}{32}x(k-x), & 0 \leq x \leq k \\ 0, & \text{otherwise} \end{cases}$$

where k is a constant.

a) Show that $k = 4$. 2

$$\int_0^k \frac{3}{32} x(k-x)\,dx = 1$$

$$\frac{3}{32}\int_0^k kx - x^2\,dx = 1$$

$$\frac{3}{32}\left[\frac{kx^2}{2} - \frac{x^3}{3}\right]_0^k = 1$$

$$\frac{3}{32}\left(\frac{k^3}{2} - \frac{k^3}{3}\right) = 1$$

$$\frac{k^3}{6} = \frac{32}{3}$$

$$k = 4$$

b) Find the cumulative distribution function $F(x)$. 2

$$F(x) = \int_0^x \frac{3}{32} x(4-x)\,dx$$

$$= \frac{3}{32}\int_0^x 4x - x^2\,dx$$

$$= \frac{3}{32}\left[2x^2 - \frac{x^3}{3}\right]_0^x$$

$$= \frac{3}{32}\left(2x^2 - \frac{x^3}{3}\right)$$

$$= \frac{3x^2}{16} - \frac{x^3}{32}$$

c) Find the mode of the probability density function. 1

$x = 2$

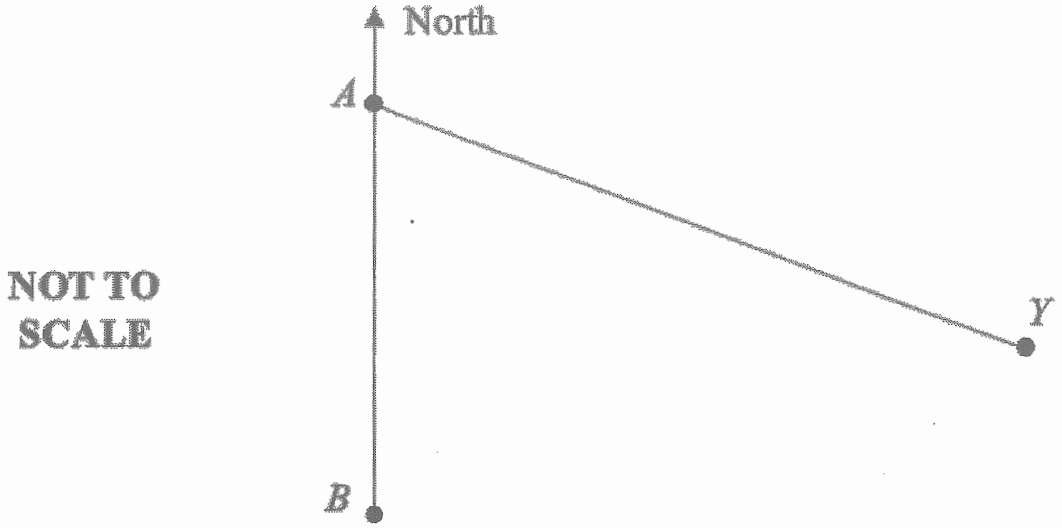
d) Find the probability that the phone will be answered within the first minute. 1

$F(1) = \frac{5}{32}$

Question Twenty-Eight

A yacht leaves Port A on a bearing of $120°$ and sails for three hours at an average speed of $15km/h$ to its destination Y where it stops.

At the same time, a speed boat also leaves from Port A and travels due south to Island B that is $30km$ from the port.



a) Calculate the distance of the yacht from Island B to the nearest kilometre. 2

Sails for 3 hours = 45km

$YB^2 = 30^2 + 45^2 - 2(30)(45)\cos 60°$

$YB \doteq 40km$

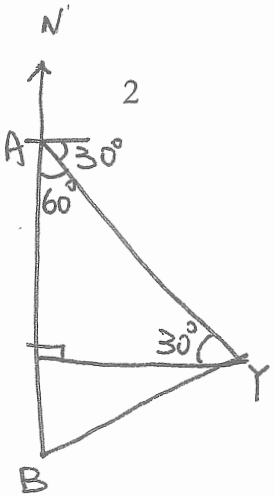
b) Find the bearing of Island B from the yacht to the nearest degree. 2

$\frac{\sin \angle YBA}{45} = \frac{\sin 60°}{YB}$

$\frac{\sin \angle YBA}{45} = \frac{\sin 60°}{40}$

$\angle YBA \doteq 79°$

So the Bearing of B from the yacht is 259°T



c) After spending several hours at the island, the speed boat travels due north back to Port A. 2

How far south of the port will the speed boat be when it is directly west of the yacht?

$\cos 60° = \frac{x}{45}$

$x = 22.5 \text{ km}$

Question Twenty-Nine

A small group of students were surveyed to determine whether they were part of the local baseball or tennis team.

- 16 people played tennis only.
- 20 people played basketball only.
- 5 people played neither.
- k people played both.

Let B be the event that a person plays baseball and let T be the event that a person plays tennis.
Determine the value of k such that the events B and T are independent. 3

B and T are independent

$P(B|T) = P(B)$

$\frac{k}{16+k} = \frac{20+k}{16+k+20+5}$

$(16+k)(20+k) = k(k+41)$

$320 + 36k + k^2 = k^2 + 41k$

$320 = 5k$

$k = 64$

Question Thirty

Packets of coffee beans are labelled with a net weight of 300g. It is found that the weight of a packet can be modelled by a normal distribution with mean 306g and standard deviation 3g.

a) Use the Empirical rule to determine the probability that the weight of one packet of coffee beans is less than the advertised weight of 300g. 1

$P(< 2\sigma \text{ below the mean}) = \frac{0.05}{2}$

$= 2.5\%$

b) In a shipment of 40 boxes, each with 100 packets of coffee beans, how many packets would be expected to be underweight? 1

$0.025 \times 40 \times 100$

$= 100$ packets

c) Manufacturers aim to ensure that the expected number of underweight packets in the shipment will be less than 20. The machine is adjusted to give a mean weight of 308g, with the standard deviation of 3g remaining the same. Will they meet their target? Justify your answer. 2

$Z = \frac{300 - 308}{3} = -2.6$

$P(Z < -2.6) = 1 - P(z < 2.6)$

$= 1 - 0.996$

$= 0.004$

$0.004 \times 4000 = 16$

So They meet their target as $16 < 20$

Question Thirty-One

A prototype rocket which is initially at rest, takes off from a launchpad on the ground. It has a time of flight of T seconds, and t is the time in seconds, where $0 \leq t \leq T$.

The velocity of the rocket, $v\ ms^{-1}$, is given by:

$$v(t) = 0.5e^t \sin\left(\frac{\pi t}{10}\right)$$

a) Shortly after the rocket takes off, the engine stops and it begins to descend towards the ground.
Find the time at which the rocket begins to descend. 2

$V = 0$, $t = ?$

$0 = 0.5e^t \sin\left(\frac{\pi t}{10}\right)$

$0.5e^t \neq 0$

$\sin\left(\frac{\pi t}{10}\right) = 0$

$\frac{\pi t}{10} = 0, \pi, 2\pi \ldots$

$t = 10$ seconds when the rocket begins to descend towards the ground.

b) Before the rocket starts to descend it reaches its maximum velocity. Find the time it takes for the rocket to achieve its maximum velocity. Give your answer correct to two decimal places. 3

for maximum velocity,

$\frac{dv}{dt} = 0$

$0.5e^t \times \frac{\pi}{10}\cos\left(\frac{\pi t}{10}\right) + \sin\left(\frac{\pi t}{10}\right) + 0.5e^t = 0$

$0 = 0.5e^t\left(\frac{\pi}{10}\cos\left(\frac{\pi t}{10}\right) + \sin\left(\frac{\pi t}{10}\right)\right)$

$0.5e^t \neq 0$

$\frac{\pi}{10}\cos\left(\frac{\pi t}{10}\right) + \sin\left(\frac{\pi t}{10}\right) = 0$

$\tan\left(\frac{\pi t}{10}\right) = -\frac{\pi}{10}$

$\frac{\pi t}{10} = -0.304, 2.837\ldots$

As $t > 0$, $t = 2.837 \times \frac{10}{\pi}$

$t \doteq 9.03$ seconds (2 decimal places)

The rocket reaches its maximum velocity at $t = 9.03$ seconds